

# Possible Winners in Noisy Elections\*

Krzysztof Wojtas Krzysztof Magiera Tomasz Miąsko Piotr Faliszewski  
AGH University of Science and Technology  
Krakow, Poland

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## Abstract

We consider the problem of predicting winners in elections, for the case where we are given complete knowledge about all possible candidates, all possible voters (together with their preferences), but where it is uncertain either which candidates exactly register for the election or which voters cast their votes. Under reasonable assumptions, our problems reduce to counting variants of election control problems. We either give polynomial-time algorithms or prove #P-completeness results for counting variants of control by adding/deleting candidates/voters for Plurality,  $k$ -Approval, Approval, Condorcet, and Maximin voting rules. We consider both the general case, where voters' preferences are unrestricted, and the case where voters' preferences are single-peaked.

## 1 Introduction

Predicting election winners is always an exciting activity: Who will be the new president? Will the company merge with another one? Will taxes be higher or lower? The goal of this paper is to establish the computational complexity of a family of problems modeling a certain type of winner-prediction problems.

Naturally, predicting winners is a hard task, full of uncertainties. For example, we typically are not sure which voters will eventually cast their votes or, sometimes, even which candidates will participate in the election (consider, e.g., a candidate withdrawing due to personal reasons). Further, typically we do not have complete knowledge regarding each voters' preferences. Nonetheless, elections are in everyday use both among humans (consider, e.g., political elections, elections among companies' shareholders, various polls on the Internet and social media, or even sport events, where judges "vote" on who is the best competitor) and among software agents (see, e.g., election applications for planning in multiagent systems [14], for recommendation systems [25, 36, 48], for web-search [12], or

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natural language processing [32]) and, thus, the problem of predicting election winners is far too important to be abandoned simply because it is difficult.

In this paper, we focus on a variant of the winner-prediction problem where we have complete knowledge regarding all possible candidates and all eligible voters (including knowledge of their preferences<sup>1</sup>), but we are uncertain as to which candidates and which voters turn up for the actual election (see Section 5 for other approaches to the problem). However, modelling uncertainty regarding both the candidate set and the voter collection on one hand almost immediately leads to computationally hard problems for typical election systems, and on the other hand does not seem to be as well motivated as focusing on each of these sets separately. Thus, we consider the following two settings:

1. The set of candidates is fixed, but for each possible subset of voters we are given a probability that exactly these voters show up for the vote.
2. The set of voters is fixed, but for each possible subset of candidates we are given a probability that exactly these candidates register for the election.

The former setting, in particular, corresponds to political elections (e.g., to presidential elections), where the candidate set is typically fixed well in advance due to election rules, the set of all possible voters (i.e., the set of all citizens eligible to vote) is known, but it is not clear as to which citizens choose to cast their votes. The latter setting may occur, for example, if one considers software agents voting on a joint plan [14]. The set of agents participating in the election is typically fixed, but various variants of the plan can be put forward or dismissed dynamically. In either case, our goal is to compute each candidate’s probability of victory. However, our task would very quickly become computationally prohibitive (or, difficult to represent on a computer) if we allowed arbitrary probability distributions. Thus, we have to choose some restriction on the distributions we consider. Let us consider the following example.

Let the set  $C$  of candidates participating in the election be fixed (e.g., because the election rules force all candidates to register well in advance). We know that some set  $V$  of voters will certainly vote (e.g., because they have already voted and this information is public<sup>2</sup>). The set of voters who have not decided to vote yet is  $W$ . From some source (e.g., from prior experience) we have a probability distribution  $P$  on the number of voters from  $W$  that will participate in the election (we assume that each equal-sized subset from  $W$  is equally likely to join the election; we have no prior knowledge as to which eligible voters are more likely to vote). That is, for each  $i$ ,  $0 \leq i \leq \|W\|$ , by  $P(i)$  we denote the probability that exactly  $i$  voters from  $W$  join the election (we assume that  $P(i)$  is easily computable). By  $Q(i)$  we denote the probability that a certain designated candidate  $p$  wins provided that exactly  $i$ , randomly chosen, voters from  $W$  participate in the election. The probability that

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<sup>1</sup>Note that while full-knowledge assumption regarding voters’ preferences might seem very unrealistic, it is standard within computational social choice literature, and in our case can often be justified (e.g., election polls can provide a good approximation of such knowledge).

<sup>2</sup>Naturally, in typical political elections such information would not be public and we would have to rely on polls. On the other hand, in multiagent systems there can be cases where votes are public.

$p$  wins is given by:

$$P(0)Q(0) + P(1)Q(1) + \cdots P(\|W\|)Q(\|W\|).$$

We use this formula to compute the probability of each candidate’s victory, which gives some idea as to who is the likely winner of the election.

To compute  $Q(i)$ , we have to compute for how many subsets  $W'$  of  $W$  of size exactly  $i$  candidate  $p$  wins after adding the voters from  $W'$  to the election, and divide it by  $\binom{\|W\|}{i}$ . That is, computing  $Q(i)$  boils down to solving a counting variant of control by adding voters problem. In the decision variant of this control problem, introduced by Bartholdi, Tovey, and Trick [3], we are given an election where some voters have already registered to vote, some are not-yet-registered, and we ask if it is possible to add some number of these not-yet-registered voters (but no more than a given limit) to ensure that a designated candidate wins. In the counting variant, studied first in this paper, we ask how many ways are there to add such a group of voters.

One can do reasoning analogous to the one presented above for the case of control by deleting voters and for the case of control by adding/deleting candidates. That is, our winner prediction problems, in essence, reduce to solving counting variants of election control problems. Our goal in this paper is, thus, to study the computational complexity of these counting control problems.

Computational study of election control was initiated by Bartholdi, Tovey, and Trick [3], and in recent years was continued by a number of researchers (we discuss related work in Section 5; we also point the reader to the survey of Faliszewski, Hemaspaandra, and Hemaspaandra [18] for a more detailed discussion of the complexity of election control). However, our paper is the first one to study counting variants of control (though, as we discuss in Section 5, the papers of Walsh and Xia [52] and of Bachrach, Betzler, and Faliszewski [1] are very close in spirit).

It is well-known that election control problems tend to have fairly high worst-case complexity. Indeed, we are not aware of a single practical voting rule for which the decision variants of the four most typical election control problems (the problems of adding/deleting candidates/voters) are all polynomial-time solvable. Naturally, when a decision variant of a counting problem is NP-hard, then we cannot hope that its counting variant would be polynomial-time solvable. In effect, from the technical perspective, our research can be seen as answering the following question: For which voting rules and for which control types is the counting variant of the control problem no harder than its decision variant? From the practical perspective, it is, nonetheless, more important to simply have effective algorithms. To this end, we follow Faliszewski et al. [23] and in addition to the general setting where each voter can cast any possible vote, we study the single-peaked case, where the voting is restricted to a certain class of votes viewed as “reasonable” (intuitively, in the single-peaked case we assume that the candidates are located on a one-dimensional spectrum of possible opinions, each voter has some favorite opinion on this spectrum, and the voters form their preferences based on candidates’ distances from their favorite position). Single-peaked preferences, introduced by Black [7] over 50 years ago, are often viewed as a natural (if somewhat

simplified due to its one-dimensional nature) model of how realistic votes might look like (however, whenever one makes a statement of this form, one should keep in mind the results of Sui, Francois-Nienaber, and Boutilier [49]). It turns out that for the single-peaked case, we obtain polynomial-time algorithms for counting variants of all our control problems and all the rules that we consider (except for the Approval rule, for which we have no results in this case).

We mention that our model of winner prediction, based on counting variants of control, is similar to, though more general than, the model where we assume that each voter casts his or her vote with some probability  $p$  (universal for all the voters). We can simulate this scenario in our model by providing an appropriate function  $P$ . Specifically, if there are  $n$  voters and the probability that each particular voter  $v$  casts his or her vote is  $p$ , then the probability that exactly  $i$  voters vote is given by the binomial distribution:

$$P(i) = \binom{n}{i} p^i (1-p)^{n-i}.$$

On the other hand, our model does not capture the situation where each voter  $v$  has a possibly different probability  $p_v$  of casting a vote. We believe that this latter model deserves study as well, but we do not focus on it in this paper.

The paper is organized as follows. First, in Section 2, we formally define elections and provide brief background on complexity theory (focusing on counting problems). Then, in Section 3, we define counting variants of election control problems and prove some of their general properties. Our main results are in Section 4. There we study the complexity of Plurality,  $k$ -Approval, Approval, Condorcet rule, and Maximin rule. For each of the rules, we consider the unrestricted case and, if we obtain a hardness result, we consider the single-peaked case (except for Approval, for which we were not able to obtain results for the single-peaked case). Finally, we discuss related work in Section 5 and bring together our conclusions and discuss future work in Section 6.

## 2 Preliminaries

**Elections.** An *election*  $E$  is a pair  $(C, V)$  such that  $C$  is a finite set of candidates and  $V$  is a finite collection of voters. We typically use  $m$  to denote the number of candidates and  $n$  to denote the number of voters. Each voter has a preference order in which he or she ranks candidates, from the most desirable one to the most despised one. For example, if  $C = \{a, b, c\}$  and a voter likes  $b$  most and  $a$  least, then this voter would have preference order  $b > c > a$ . (However, under Approval voting, instead of ranking the candidates each voter simply indicates which candidates he or she approves of.) We sometimes use the following notation. Let  $A$  be some subset of the candidate set. Putting  $A$  in a preference order means listing members of  $A$  in lexicographic order and putting  $\overleftarrow{A}$  in a preference order means listing members of  $A$  in the reverse of the lexicographic order. For example, if  $C = \{a, b, c, d\}$  then  $a > C - \{a, b\} > d$  means  $a > c > d > b$  and  $a > \overleftarrow{C - \{a, b\}} > d$  means  $a > d > c > b$ . Given an election  $E = (C, V)$ , we write  $N_E(c, c')$  to denote the number of voters in  $V$  that

prefer  $c$  to  $c'$ . The function  $N_E$  is sometimes referred to as the *weighted majority graph* of election  $E$ .

In general, given a candidate set  $C$ , the voters are free to report any preference order over  $C$  (we refer to this setting as the *unrestricted case*). However, it is often more realistic to assume some restriction on the domain of possible votes (for example, in real-life political elections we do not expect to see many voters who rank the extreme left-wing candidate first, then the extreme right-wing candidate, and so on, in an interleaving fashion). Thus, in addition to the unrestricted case, we also study the case of *single-peaked* preferences of Black [7] (this domain restriction is quite popular in the computational social choice literature, and was already studied, e.g., by Conitzer [11], Walsh [51], Faliszewski et al. [23], and others; see Section 5).

**Definition 2.1.** *Let  $C$  be a set of candidates and let  $L$  be a linear order over  $C$  (we refer to  $L$  as the societal axis). We say that a preference order  $>$  (over  $C$ ) is single-peaked with respect to  $L$  if for each three candidate  $a, b, c \in C$ , it holds that*

$$(a \ L \ b \ L \ c \vee c \ L \ b \ L \ a) \implies (a > b \implies b > c)$$

*An election  $E = (C, V)$  is single-peaked with respect to  $L$  if each vote in  $V$  is single-peaked with respect to  $L$ . An election  $E = (C, V)$  is single-peaked if there is a societal axis  $L$  such that  $E$  is single-peaked with respect to  $L$ .*

There are polynomial-time algorithms that given an election  $E$  verify if it is single-peaked and, if so, provide the appropriate societal axis witnessing this fact [4, 17, 2]. Thus, following Walsh [51], whenever we consider problems about single-peaked elections, we assume that we are given appropriate societal axis as part of the input (if it were not provided, we could always compute it.)

**Voting Systems.** A *voting system* (a *voting rule*) is a rule which specifies how election winners are determined. We allow an election to have more than one winner, or even to not have winners at all. In practice, tie-breaking rules are used, but here we disregard this issue by simply using the unique winner model (see Definition 3.1). However, we point the reader to [42, 41, 43] for a discussion regarding the influence of tie-breaking for the case of election manipulation problems (see [18, 24, 9] for overviews of the manipulation problem specifically and computational aspects of voting generally).

We consider the following voting rules (in the description below we assume that  $E = (C, V)$  is an election with  $m$  candidates and  $n$  voters):

**Plurality.** Under Plurality, each candidate receives a point for each vote where this candidate is ranked first. The candidates with most points win.

**$k$ -Approval.** For each candidate  $c \in C$ , we define  $c$ 's  $k$ -Approval score,  $score_E^k(c)$ , to be the number of voters in  $V$  that rank  $c$  among the top  $k$  candidates; the candidates with highest scores win. Note that Plurality is, in effect, a nickname for 1-Approval. (We mention that  $k$ -Veto means  $(m - k)$ -Approval, though we do not study  $k$ -Veto in this paper).

**Approval.** Under Approval (without the qualifying “ $k$ -”), the score of a candidate  $c \in C$ ,  $score_E^a(c)$ , is the number of voters that approve of  $c$  (recall that under Approval the voters do not cast preference orders but 0/1 approval vectors, where for each candidate they indicate if they approve of this candidate or not). Again, the candidates with highest scores win. (Note that the notion of single-peaked elections that we have provided as Definition 2.1 does not apply to preferences specified as approval vectors; Faliszewski et al. [23] have provided a natural variant of single-peakedness for approval vectors, but since we did not obtain results for this case, we omitted this definition.)

**Condorcet.** A candidate  $c$  is a *Condorcet winner* exactly if  $N_E(c, c') > N_E(c', c)$  for each  $c' \in C - \{c\}$ . A candidate  $c$  is a *weak Condorcet winner* exactly if  $N_E(c, c') \geq N_E(c', c)$  for each  $c' \in C - \{c\}$ . Note that if an election has a Condorcet winner, then this winner is unique (though, an election may have several weak Condorcet winners).<sup>3</sup> We recall the classic result that if the voters are single-peaked then the election always has (weak) Condorcet winner(s) (if the number of voters is odd, then the election has a unique Condorcet winner).

**Maximin.** Maximin rule is defined as follows. The score of a candidate  $c$  is defined as  $score_E^m(c) = \min_{d \in C - \{c\}} N_E(c, d)$ . The candidates with highest Maximin score are Maximin winners.

Maximin is an example of a so-called Condorcet-consistent rule. A rule  $R$  is Condorcet-consistent if the following holds: If  $E$  is an election with (weak) Condorcet winner(s), then the  $R$ -winners of  $E$  are exactly these (weak) Condorcet winner(s). We also considered studying the Copeland rule (see, e.g., the work of Faliszewski et al. [22] regarding the complexity of control for the Copeland rule), but for this rule all relevant types of control are NP-complete and, so, at best we would obtain #P-completeness results based on simple generalizations of proofs from the literature. Nonetheless, since Copeland’s rule is Condorcet-consistent, results for the single-peaked case are valid for it.

**Notation for Graphs.** We assume familiarity with basic concepts of graph theory. Given an undirected graph  $G$ , by  $V(G)$  we mean its set of vertices, and by  $E(G)$  we mean its set of edges.<sup>4</sup> Whenever we discuss a bipartite graph  $G$ , we assume that  $V(G)$  is partitioned into two subsets,  $X$  and  $Y$ , such that each edge connects some vertex from  $X$  with some vertex from  $Y$ . We write  $X(G)$  to denote  $X$ , and  $Y(G)$  to denote  $Y$ .

**Computational Complexity.** We assume that the reader is familiar with standard notions of computational complexity theory, as presented, for example, in the textbook of

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<sup>3</sup>We mention that sometimes Condorcet’s rule is not considered a voting rule (but, for example, a consensus notion [39]) because there are elections for which there are no Condorcet winners. However, Condorcet’s rule has been traditionally studied in the context of election control, and—thus—we study it to (a) enable comparison with other papers, and (b) because for single-peaked elections the results for Condorcet rule translate to all Condorcet-consistent rules.

<sup>4</sup>We use this slightly nonstandard notation to be able to also use symbols  $V$  and  $E$  to denote voter collections and elections.

Papadimitriou [44], but we briefly review notions regarding the complexity theory of counting problems. Let  $A$  be some computational problem where for each instance  $I$  we ask if there exists some mathematical object satisfying a given condition. In the counting variant of  $A$ , denoted  $\#A$ , for each instance  $I$  we ask for the number—denoted  $\#A(I)$ —of the objects that satisfy the condition. For example, consider the following problem.

**Definition 2.2.** *An instance of  $X3C$  is a pair  $(B, \mathcal{S})$ , where  $B = \{b_1, \dots, b_{3k}\}$  and  $\mathcal{S} = \{S_1, \dots, S_n\}$  is a family of 3-element subsets of  $B$ . In  $X3C$  we ask if it is possible to find exactly  $k$  sets in  $\mathcal{S}$  whose union is exactly  $B$ . In  $\#X3C$  we ask how many  $k$ -element subsets of  $\mathcal{S}$  have  $B$  as their union.*

The class of counting variants of NP-problems is called  $\#P$  and the class of functions computable in polynomial time is called  $FP$ . To reduce counting problems to each other, we will use one of the following reducibility notions.

**Definition 2.3.** *Let  $\#A$  and  $\#B$  be two counting problems.*

1. *We say that  $\#A$  Turing reduces to  $\#B$  if there exists an algorithm that solves  $\#A$  in polynomial time given oracle access to  $\#B$  (i.e., given the ability to solve  $\#B$  instances in unit time).*
2. *We say that  $\#A$  metric reduces to  $\#B$  if there exist two polynomial-time computable functions,  $f$  and  $g$ , such that for each instance  $I$  of  $\#A$  it holds that (1)  $f(I)$  is an instance of  $\#B$ , and (2)  $\#A(I) = g(I, \#B(f(I)))$ .*
3. *We say that  $\#A$  parsimoniously reduces to  $\#B$  if  $\#A$  metric reduces to  $\#B$  via functions  $f$  and  $g$  such that for each instance  $I$  and each integer  $k$ ,  $g(I, k) = k$  (i.e., there is a polynomial-time computable function  $f$  such that for each instance  $I$  of  $\#A$ , we have  $\#A(I) = \#B(f(I))$ ).*

It is well-known that these reducibility notions are transitive. For a given reducibility notion  $R$ , we say that a problem is  $\#P$ - $R$ -complete if it belongs to  $\#P$  and every  $\#P$ -problem  $R$  reduces to it. For example,  $\#X3C$  is  $\#P$ -parsimonious-complete [29]. Throughout this paper we will write  $\#P$ -complete to mean  $\#P$ -parsimonious-complete. Turing reductions were used, e.g., by Valiant [50] to show  $\#P$ -hardness of computing a permanent of a 0/1 matrix. As a result, he also showed  $\#P$ -Turing-hardness of the following problem.

**Definition 2.4.** *In  $\#PerfectMatching$  we are given a bipartite graph  $G = (G(X), G(Y), G(E))$  with  $\|G(X)\| = \|G(Y)\|$  and we ask how many perfect matchings does  $G$  have.*

Zankó [55] improved upon this result by showing that  $\#PerfectMatching$  is  $\#P$ -many-one-complete (“many-one” is yet another reducibility type, more general than parsimonious reductions but less general than metric reductions). From our perspective, it suffices that, in effect,  $\#PerfectMatching$  is  $\#P$ -metric-complete. Metric reductions were introduced by Krentel [31], and parsimonious reductions were defined by Simon [47].

### 3 Counting Variants of Control Problems

In this section we formally define counting variants of election control problems and show some interconnections between some of them. We are interested in control by adding candidates (AC), control by deleting candidates (DC), control by adding voters (AV), and control by deleting voters (DV). For each of these problems, we consider its constructive variant (CC) and its destructive variant (DC).

**Definition 3.1.** *Let  $R$  be a voting system. In each of the counting variants of constructive control problems we are given a candidate set  $C$ , a voter collection  $V$ , a nonnegative integer  $k$ , and a designated candidate  $p \in C$ . In constructive control by adding voters we are additionally given a collection  $W$  of unregistered voters, and in constructive control by adding candidates we are additionally given a set  $A$  of unregistered candidates. In these problems we ask for the following quantities:*

1. *In control by adding voters ( $R$ -#CCAV), we ask how many sets  $W'$ ,  $W' \subseteq W$ , are there such that  $p$  is the unique winner of  $R$ -election  $(C, V \cup W')$ , where  $\|W'\| \leq k$ .*
2. *In control by deleting voters ( $R$ -#CCDV), we ask how many sets  $V'$ ,  $V' \subseteq V$  are there such that  $p$  is the unique winner of  $R$ -election  $(C, V - V')$ , where  $\|V'\| \leq k$ .*
3. *In control by adding candidates ( $R$ -#CCAC), we ask how many sets  $A'$ ,  $A' \subseteq A$ , are there such that  $p$  is the unique winner of  $R$ -election  $(C \cup A', V)$ , where  $\|A'\| \leq k$ .*
4. *In control by deleting candidates ( $R$ -#CCDC), we ask how many sets  $C'$ ,  $C' \subseteq C$ , are there such that  $p$  is the unique winner of  $R$ -election  $(C - C', V)$ , where  $\|C'\| \leq k$  and  $p \notin C'$ .*

*Destructive variants are defined identically, except that we ask for the number of settings where the designated candidate is not the unique winner.*

(To obtain decision variants of the above problems, simply change the question from asking for a particular quantity to asking if that quantity is greater than zero. We mention that in the literature researchers often study both the unique-winner model—as we do here—and the nonunique-winner model, where it suffices to be one of the winners. For the rules that we study, results for both models are the same.)

The above problems are interesting for several reasons. First, as described in the introduction, we believe that they are useful as models for various scenarios pertaining to predicting election winners. Second, they are counting variants of the well-studied control problems [3, 28]. Third, they expose intuitive connections between various types of control. In particular, we have the following results linking the complexity of destructive variants with that of the constructive ones, and linking the complexity of the “deleting” variants with that of the “adding” variants.

**Theorem 3.2.** *Let  $R$  be a voting system, let  $\#C$  be one of  $R$ -#CCAC,  $R$ -#CCDC,  $R$ -#CCAV,  $R$ -#CCDV, and let  $\#D$  be the destructive variant of  $\#C$ . Then,  $\#C$  metric reduces to  $\#D$  and  $\#D$  metric reduces to  $\#C$ .*



*Proof.* We give a metric reduction from  $\#\mathcal{C}$  to  $\#\mathcal{D}$ . Let  $I$  be an instance of  $\#\mathcal{C}$ , where the goal is to make some candidate  $p$  the unique winner. We define  $f(I)$  to be an instance of  $\#\mathcal{D}$  that is identical to  $I$ , except that the goal is to ensure that  $p$  is not the unique winner. Let  $s_I$  be the total number of control actions allowed in  $I$ <sup>5</sup> (and, naturally, also the total number of control actions allowed in  $f(I)$ ). It is easy to see that  $s_I$  is polynomial-time computable and that the number of solutions for  $I$  is exactly  $s_I - \#\mathcal{D}(f(I))$ . Thus, we define  $g(I, \#\mathcal{D}(f(I))) = s_I - \#\mathcal{D}(f(I))$ . We see that the reduction is polynomial-time and correct. The same argument shows that  $\#\mathcal{D}$  metric reduces to  $\#\mathcal{C}$ .  $\square$

**Theorem 3.3.** *Let  $R$  be a voting system, then  $R$ - $\#\text{CCDV}$  Turing reduces to  $R$ - $\#\text{CCAV}$  and  $R$ - $\#\text{CCDC}$  Turing reduces to  $R$ - $\#\text{CCAC}$ .*

*Proof.* Let us fix a voting system  $R$ . The proofs that  $R$ - $\#\text{CCDV}$  Turing reduces to  $R$ - $\#\text{CCAV}$  and that  $R$ - $\#\text{CCDC}$  Turing-reduces to  $R$ - $\#\text{CCAV}$  are very similar and thus we discuss them jointly. Let  $I = (C, V, p, k)$  be an input instance of  $R$ - $\#\text{CCDV}$  (of  $R$ - $\#\text{CCDC}$ ), where  $C$  is the candidate set,  $V$  is the collection of voters,  $p$  is the designated candidate, and  $k$  is the upper bound on the number of voters that can be removed. We define the following transformations of  $I$ :

1. For the case of  $R$ - $\#\text{CCDV}$ , for each nonnegative integer  $g$ ,  $g \leq \|V\|$ , let  $J_g$  be an instance of  $R$ - $\#\text{CCAV}$ , where the candidate set is  $C$ , the set of registered voters is empty, the collection of unregistered voters is  $V$ , the designated candidate is  $p$ , and the bound on the number of voters that can be added is  $g$ . That is,  $J_g = (C, \emptyset, V, p, g)$ .
2. For the case of  $R$ - $\#\text{CCDC}$ , for each nonnegative integer  $g$ ,  $g \leq \|C\| - 1$ , let  $J_g$  be an instance of  $R$ - $\#\text{CCAC}$ , where the candidate set is  $\{p\}$ , the set of registered voters is  $V$ , the set of unregistered candidates is  $C - \{p\}$ , the designated candidate is  $p$ , and the bound on the number of candidates that can be added is  $g$ . That is,  $J_g = (\{p\}, C - \{p\}, V, p, g)$ .

To complete the reduction, it suffices to note that for the case of  $R$ - $\#\text{CCDV}$  it holds that:

$$R\text{-}\#\text{CCDV}(I) = \begin{cases} R\text{-}\#\text{CCAV}(J_{\|V\|}) & , \text{ if } k \geq \|V\| \\ R\text{-}\#\text{CCAV}(J_{\|V\|}) - R\text{-}\#\text{CCAV}(J_{\|V\|-k-1}) & , \text{ if } 0 \leq k < \|V\| \end{cases}$$

and for the case of  $R$ - $\#\text{CCDC}$  it holds that:

$$R\text{-}\#\text{CCDC}(I) = \begin{cases} R\text{-}\#\text{CCAC}(J_{\|C\|-1}) & , \text{ if } k \geq \|C\| - 1 \\ R\text{-}\#\text{CCAC}(J_{\|C\|-1}) - R\text{-}\#\text{CCAC}(J_{\|C\|-k-2}) & , \text{ if } 0 \leq k < \|C\| - 1 \end{cases}$$

These expressions define a natural algorithm for solving  $R$ - $\#\text{CCDV}$  ( $R$ - $\#\text{CCDC}$ ) given at most two calls to an  $R$ - $\#\text{CCAV}$  (an  $R$ - $\#\text{CCAC}$ ) oracle. It is clear that this Turing reduction runs in polynomial time.  $\square$

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<sup>5</sup>For example, if  $\#\mathcal{C}$  was  $\#\text{CCDV}$  and  $I = (C, V, p, k)$ , then  $s_I$  would be the number of up-to-size- $k$  subsets of  $V$  (i.e., the number of subsets of voters that can be deleted from  $V$ ).

Theorems 3.2 and 3.3 are very useful and, in particular, we obtain all of our destructive-case hardness results via Theorem 3.2, and all our “deleting”-case easiness results using Theorem 3.3.

Obtaining results similar to Theorems 3.2 and 3.3 for the decision variants of control problems is more difficult. Nonetheless, recently several researchers have made some progress on this front. In particular, Hemaspaandra, Hemaspaandra, and Menton [27] have shown that control by partition of candidates and control by runoff partition of candidates (two types of control not discussed in this paper) are equivalent in terms of their computational complexity. Focusing on particular classes of voting rules, for the case of  $k$ -Approval and  $k$ -Veto, Faliszewski, Hemaspaandra, and Hemaspaandra [20] have shown that control by deleting voters reduces to control by adding voters, and Miąsko [40] achieved the same for the case of voting rules based on the weighted majority graphs (e.g., for Borda, Condorcet, Maximin, and Copeland).<sup>6</sup>

## 4 Results

We now present our complexity results regarding counting variants of election control problems. We consider two cases: the unrestricted case, where the voters can have any arbitrary preference orders, and the single-peaked case, where voters’ preference orders are single-peaked with respect to a given societal axis. We summarize our results in Table 1. For comparison, in Table 2 we quote results for the decision variants of the respective control problems.

Let us consider the unrestricted case first. Not surprisingly, for each of our control problems whose decision variant is NP-complete, the counting variant is #P-complete for some reducibility type. However, interestingly, we have also found examples of election systems and control types where the decision variant is easy, but the counting variant is hard. For example, for 2-Approval we have that all decision variants of voter control problems are in P [33, 20], yet all their counting variants are #P-Turing-complete. Similarly, for Maximin all decision variants of candidate control (except constructive control by adding candidates) are in P [19], yet their counting variants are #P-Turing-complete (or are complete for #P through even less demanding reducibility types).

The situation for the single-peaked case is quite interesting as well. While we found polynomial-time algorithms for all of our control problems for Plurality,  $k$ -Approval, and all Condorcet-consistent rules, we did not obtain results for voter control under Approval (candidate control for Approval is easy even in the unrestricted case). Decision variants of voter control are NP-complete for Approval in the unrestricted case, but—very interestingly—are in P for the single-peaked case. However, the algorithm for the single-peaked case is a clever greedy approach that seems to be very difficult to adapt to the counting case. Further, as opposed to the  $k$ -Approval case, for voter control under Approval, there does not seem to be

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<sup>6</sup>Miąsko’s master’s thesis gives this result for the case of Borda and destructive control only, but it is clear that the proof technique adapts to constructive control and that it applies to all voting rules based on weighted majority graphs.

(a) The Unrestricted Case

Problem	Plurality	$k$ -Approval, $k \geq 2$	Approval	Condorcet	Maximin
#CCAC	#P-com.	#P-com.	–	–	#P-com.
#DCAC	#P-metric-com.	#P-metric-com.	FP	FP	#P-metric-com.
#CCDC	#P-com.	#P-com.	FP	FP	#P-Turing-com.
#DCDC	#P-metric-com.	#P-metric-com.	–	–	#P-Turing-com.
#CCAV	FP	#P-Turing-com.	#P-com.	#P-com.	#P-com.
#DCAV	FP	#P-Turing-com.	#P-metric-com.	#P-metric-com.	#P-metric-com.
#CCDV	FP	#P-metric-com.	#P-com.	#P-com.	#P-com.
#DCDV	FP	#P-metric-com.	#P-metric-com.	#P-metric-com.	#P-metric-com.

(b) The Single-Peaked Case

Problem	Plurality	$k$ -Approval, $k \geq 2$	Approval	Condorcet-Consistent (e.g., Maximin)
#CCAC	FP	FP	–	–
#DCAC	FP	FP	FP	FP
#CCDC	FP	FP	FP	FP
#DCDC	FP	FP	–	–
#CCAV	FP	FP	?	FP
#DCAV	FP	FP	?	FP
#CCDV	FP	FP	?	FP
#DCDV	FP	FP	?	FP

Table 1: The complexity of counting variants of control problems for (a) the unrestricted case, and for (b) the single-peaked case. A dash in an entry means that the given system is *immune* to the type of control in question (i.e., it is impossible to achieve the desired effect by the action this control problem allows; technically this means the answer to the counting question is always 0). Immunity results were established by Bartholdi, Tovey, and Trick (1989) for the constructive cases, and by Hemaspaandra, Hemaspaandra, and Rothe (2007) for the destructive cases.

a natural dynamic-programming-based approach. Naturally, we also tried to find a hardness proof, but we failed at that as well because the problem seems to have quite a rich structure. (However, see the work of Miąsko [40] for the case of priced control under Approval with single-peaked preferences.)

In the following sections we give proofs for our results and provide some more detailed discussion regarding particular voting rules.

(a) The Unrestricted Case

Problem	Plurality	2-Approval	3-Approval	$k$ -Approval, $k \geq 4$	Approval	Condorcet	Maximin
CCAC	NP-com.	NP-com.	NP-com.	NP-com.	–	–	NP-com.
DCAC	NP-com.	NP-com.	NP-com.	NP-com.	P	P	P
CCDC	NP-com.	NP-com.	NP-com.	NP-com.	P	P	P
DCDC	NP-com.	NP-com.	NP-com.	NP-com.	–	–	P
CCAV	P	P	P	NP-com.	NP-com.	NP-com.	NP-com.
DCAV	P	P	P	NP-com.	P	P	NP-com.
CCDV	P	P	NP-com.	NP-com.	NP-com.	NP-com.	NP-com.
DCDV	P	P	P	NP-com.	P	P	NP-com.

(b) The Single-Peaked Case

Problem	Plurality	$k$ -Approval, $k \geq 2$	Approval	Condorcet-Consistent (e.g., Maximin)
CCAC	P	P	–	–
DCAC	P	P	P	P
CCDC	P	P	P	P
DCDC	P	P	–	–
CCAV	P	P	P	P
DCAV	P	P	P	P
CCDV	P	P	P	P
DCDV	P	P	P	P

Table 2: The complexity of decision variants of control problems for (a) the unrestricted case, and for (b) the single-peaked case. A dash in an entry means that the given system is *immune* to the type of control in question. For the unrestricted case, we have the following: The results for Plurality are due to Bartholdi, Tovey, and Trick [3] and Hemaspaandra, Hemaspaandra, and Rothe [28], the results regarding  $k$ -Approval are due to Lin [33] (see also [13, 20]), the results regarding Approval are due to Hemaspaandra, Hemaspaandra, and Rothe [28], and the results regarding Maximin are due to Faliszewski, Hemaspaandra, and Hemaspaandra [19]. For the single-peaked case, we inherit polynomial-time algorithms from the unrestricted case. The remaining results (except those for Condorcet-consistent rules) are due to Faliszewski et al. [23], and the remaining results regarding Condorcet-consistent rules are due to [8].

#### 4.1 Plurality Voting

Under plurality voting, counting variants of both control by adding voters and control by deleting voters are in FP even for the unrestricted case. Our algorithms are based on

dynamic programming and applications of Theorems 3.2 and 3.3.

**Theorem 4.1.** *Plurality-#CCAV, Plurality-#DCAV, Plurality-#CCDV, and Plurality-#DCDV are in FP, even in the unrestricted case.*

*Proof.* We give the proof for Plurality-#CCAV only. The result for Plurality-#CCDV follows by applying Theorem 3.3, and the destructive cases follows by applying Theorem 3.2.

Let  $I = (C, V, W, p, k)$  be an input instance of Plurality-#CCAV, where  $C = \{p, c_1, \dots, c_{m-1}\}$  is the candidate set,  $V$  is the set of registered voters,  $W$  is the set of unregistered voters,  $p$  is the designated candidate, and  $k$  is the upper bound on the number of voters that can be added. We now describe a polynomial-time algorithm that computes the number of solutions for  $I$ .

Let  $A_p$  be the set of voters from  $W$  that rank  $p$  first. Similarly, for each  $c_i \in C$ , let  $A_{c_i}$  be the set of voters from  $W$  that rank  $c_i$  first. We also define  $\text{count}(C, V, W, p, k, j)$  to be the number of sets  $W' \subseteq W - A_p$  such that (1)  $\|W'\| \leq k - j$ , and (2) in election  $(C, V \cup W')$  each candidate  $c_i \in C$ ,  $1 \leq i \leq m - 1$ , has score at most  $\text{score}_{(C,V)}^p(p) + j - 1$ .

Our algorithm works as follows. First, we compute  $k_0$ , the minimum number of voters from  $A_p$  that need to be added to  $V$  to ensure that  $p$  has plurality score higher than any other candidate (provided no other voters are added). Clearly, if  $p$  already is the unique winner of  $(C, V)$  then  $k_0$  is 0, and otherwise  $k_0$  is  $\max_{c_i \in C} (\text{score}_{(C,V)}^p(c_i) - \text{score}_{(C,V)}^p(p) + 1)$ . Then, for each  $j$ ,  $k_0 \leq j \leq \min(k, \|A_p\|)$ , we compute the number of sets  $W'$ ,  $W' \subseteq W$ , such that  $W'$  contains exactly  $j$  voters from  $A_p$ , at most  $k - j$  voters from  $W - A_p$ , and  $p$  is the unique winner of  $(C, V \cup W')$ . It is easy to verify that for a given  $j$ , there is exactly  $h(j) = \binom{\|A_p\|}{j} \cdot \text{count}(C, V, W, p, k, j)$  such sets. Our algorithm returns  $\sum_{j=k_0}^{\min(k, \|A_p\|)} h(j)$ . The reader can easily verify that this indeed is the correct answer. To complete the proof it suffices to show a polynomial-time algorithm for computing  $\text{count}(C, V, W, p, k, j)$ .

Let us fix  $j$ ,  $k_0 \leq j \leq \min(k, \|A_p\|)$ . We now show how to compute  $\text{count}(C, V, W, p, k, j)$ . Our goal is to count the number of ways in which we can add at most  $k - j$  voters from  $W - A_p$  so that no candidate  $c_i \in C$  has score higher than  $\text{score}_{(C,V)}^p(p) + j - 1$ . For each candidate  $c_i \in C$ , we can add at most  $l_i = \min(\|A_{c_i}\|, j + \text{score}_{(C,V)}^p(p) - \text{score}_{(C,V)}^p(c_i) - 1)$  voters from  $A_{c_i}$ ; otherwise  $c_i$ 's score would exceed  $\text{score}_{(C,V)}^p(p) + j - 1$ .

For each  $i$ ,  $1 \leq i \leq m - 1$ , and each  $t$ ,  $0 \leq t \leq k - j$ , let  $a_{t,i}$  be the number of sets  $W' \subseteq A_{c_1} \cup A_{c_2} \cup \dots \cup A_{c_i}$  that contain exactly  $t$  voters and such that each candidate  $c_1, c_2, \dots, c_i$  has score at most  $\text{score}_{(C,V)}^p(p) + j - 1$  in the election  $(C, V \cup W')$ . Naturally,  $\text{count}(C, V, W, p, k, j) = \sum_{t=0}^{k-j} a_{t,m-1}$ . It is easy to check that  $a_{t,i}$  satisfies the following recursion:

$$a_{t,i} = \begin{cases} \sum_{s=0}^{\min(l_i, t)} \binom{\|A_{c_i}\|}{s} a_{t-s, i-1}, & \text{if } t > 0, i > 1, \\ 1, & \text{if } t = 0, i > 1, \\ \binom{\|A_{c_1}\|}{t}, & \text{if } t \leq \|A_{c_1}\|, i = 1, \\ 0, & \text{if } t > \|A_{c_1}\|, i = 1. \end{cases}$$

Thus, for each  $t, i$  we can compute  $a_{t,i}$  using standard dynamic programming techniques in polynomial time. Thus,  $\text{count}(C, V, W, p, k, j)$  also is polynomial-time computable. This completes the proof that Plurality-#CCAV is in FP.  $\square$

On the other hand, for Plurality voting #CCAC and #CCDC are #P-complete and this follows from proofs already given in the literature [21].

**Theorem 4.2.** *In the unrestricted case, Plurality-#CCAC and Plurality-#CCDC are #P-complete.*

*Proof.* Faliszewski, Hemaspaandra, and Hemaspaandra [21, Theorem 6.4] show that decision variants of control by adding candidates and control by deleting candidates are NP-complete.<sup>7</sup> Their proofs work by reducing X3C to appropriate control problems in a parsimonious way. This means that the same reductions reduce #X3C to the counting variants of respective control problems.  $\square$

Now, Corollary 4.3 follows by combining Theorems 4.2 and 3.2.

**Corollary 4.3.** *In the unrestricted case, Plurality-#DCAC and Plurality-#DCDC are #P-metric-complete.*

However, for the single-peaked case, we get easiness results.

**Theorem 4.4.** *For the case of single-peaked profiles, Plurality-#CCAC, Plurality-#CCDC, Plurality-#DCAC, and Plurality-#DCDC are in FP.*

*Proof.* This result follows from Theorem 4.6 (see the next section) for the case of 1-Approval.  $\square$

## 4.2 $k$ -Approval Voting

While  $k$ -Approval is in many respects a simple generalization of the Plurality rule, it turns out that for  $k \geq 2$ , for the unrestricted case, all counting variants of control problems are intractable for  $k$ -Approval. This is quite expected for candidate control because decision variants of these problems are NP-complete (see the work of Lin [33, 13] and additionally of Faliszewski, Hemaspaandra, and Hemaspaandra [20]), but is more intriguing for voter control (as shown by Lin, for 2-Approval all voter control decision problems are in P, and, as one can verify, for  $k$ -Approval all destructive voter control decision problems are in P). On the other hand, for the single-peaked case all the counting variants of control are polynomial-time solvable for  $k$ -Approval (however, note that we rely on  $k$  being a constant; our results do not generalize to the case where  $k$  is part of the input).

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<sup>7</sup>Naturally, these results are originally due to Bartholid, Tovey, and Trick [3]. However, the proofs of Faliszewski, Hemaspaandra, and Hemaspaandra are more useful in our case, because they work for  $k$ -Approval for each fixed  $k$  (which will be useful later), and it is convenient to verify that they give parsimonious reductions from a #P-complete problem.

**Candidate Control.** We start by quickly dealing with candidate control in the unrestricted case. For the unrestricted case, as in the case of Plurality, we can use the result of Faliszewski, Hemaspaandra, and Hemaspaandra [21].

**Theorem 4.5.** *In the unrestricted case, for each  $k$ ,  $k \geq 2$ ,  $k$ -Approval-#CCAC and  $k$ -Approval-#CCDC are #P-complete, and  $k$ -Approval-#DCAC and  $k$ -Approval-#DCDC are #P-metric-complete.*

*Proof.* For the constructive variants, the same approach as in the proof of Theorem 4.2 works: The hardness proofs given by Faliszewski, Hemaspaandra, and Hemaspaandra [21, Theorem 6.4] apply to the case of  $k$ -Approval as well. For the destructive variants, we invoke Theorem 3.2.  $\square$

For the single-peaked case, we use a dynamic-programming approach. Our algorithm is a very extensive generalization of the algorithm for the single-peaked variant of Plurality-CCAC given by Faliszewski et al. [23]

**Theorem 4.6.** *For the case of single-peaked profiles, for each fixed positive  $k$ ,  $k$ -Approval-#CCAC,  $k$ -Approval-#CCDC,  $k$ -Approval-#DCAC, and  $k$ -Approval-#DCDC are in FP.*

*Proof.* We give the proof for  $k$ -Approval-#CCAC only. The result for  $k$ -Approval-#CCDC follows by applying Theorem 3.3, and the destructive cases follow by applying Theorem 3.2.

We are given a candidate set  $C$ , a voter collection  $V$ , a designated candidate  $p \in C$ , a collection  $A$  of unregistered candidates, a positive integer  $\ell$ , and a societal axis  $L$  over  $C \cup A$ . We show a polynomial time algorithm for determining the number of sets  $A'$ ,  $A' \subseteq A$ , where  $\|A'\| \leq \ell$ , such that  $p$  is the unique  $k$ -Approval winner in election  $(C \cup A', V)$ . We assume that  $\|C \cup A\| > 4k$  (otherwise we solve our problem by enumerating all possible solutions).

Intuitively, the idea of our algorithm is as follows: First, we fix two groups of  $k$  candidates, one “to the left” of  $p$  (with respect to  $L$ ) and one “to the right” of  $p$ . We will argue that each choice of such “neighborhood” of  $p$  fixes  $p$ ’s score. Second, with  $p$ ’s score fixed, we count how many ways are there to add candidates so that every candidate has fewer points than  $p$ . We sum these values over all choices of  $p$ ’s “neighborhood.”

Let us introduce some useful notation. For each set  $X \subseteq C \cup A$  of candidates we define  $\text{before}(X)$  to be the subset of those candidates in  $C \cup A$  that precede each member of  $X$  with respect to  $L$  (i.e.,  $\text{before}(X) = \{d \in C \cup A \mid (\forall x \in X)[d L x]\}$ ). Analogously, we define  $\text{after}(X)$  to be the subset of those candidates in  $C \cup A$  that succeed all members of  $X$  with respect to  $L$  (i.e.,  $\text{after}(X) = \{d \in C \cup A \mid (\forall x \in X)[x L d]\}$ ). The next lemma describes how we can fix candidates’ scores by fixing their “neighborhoods.”

**Lemma 4.7.** *For each set  $H \subseteq C \cup A$ ,  $\|H\| = k$ , each set  $X \subseteq \text{before}(H)$  and each candidate  $c \in X$ , if  $z$  is the score of candidate  $c$  in  $k$ -Approval election  $(X \cup H, V)$ , then for each  $Y \subseteq \text{after}(H)$  in  $k$ -Approval election  $(X \cup H \cup Y, V)$  the score of candidate  $c$  is  $z$  as well.*

*Proof.* For a given subset  $Y \subseteq \text{after}(H)$  and candidate  $c \in X$ , suppose that the score of candidate  $c$  in  $k$ -Approval election  $(X \cup H \cup Y, V)$  is  $z' \neq z$ . Adding a candidate can never increase the score of a candidate already present, so we conclude that  $z' < z$ . It means that there is a voter  $v \in V$  from which  $c$  gains a point in election  $E_1 = (X \cup H, V)$  but not in election  $E_2 = (X \cup H \cup Y, V)$ . Thus there is a candidate  $c' \in Y$  that receives a point from  $v$ . Since  $c$  is among  $v$ 's top  $k$  most preferred candidates across  $X \cup H$  and  $\|H\| = k$ , there must exist a candidate  $c'' \in H$  that is less preferred than  $c$  by  $v$ . Since  $v$  prefers  $c'$  over  $c$ , it must be that  $v$  prefers  $c'$  over  $c''$ . However, we know that  $c \succ_L c'' \succ_L c'$ , because  $c \in X$ ,  $c'' \in H$  and  $c' \in Y$ . Since  $v$  is single-peaked with respect to  $L$ ,  $v$  cannot prefer both  $c$  and  $c'$  over  $c''$ ; a contradiction.  $\square$

Let  $B_L$  be the set of the first  $2k$  candidates from  $C \cup A$  (with respect to  $L$ ) and let  $B_R$  be the set of the last  $2k$  candidates from  $C \cup A$  (with respect to  $L$ ). Without loss of generality, we assume that  $B_L$  and  $B_R$  contain candidates from  $C$  only, and that each voter has a preference order of the form  $C - (B_L \cup B_R) > B_L > B_R$  (if this were not the case then we could add to  $C$  two groups of  $2k$  dummy candidates that all the voters rank last, respecting the above requirement, and that are at the extreme ends of the societal axis  $L$ ). Note that by our assumptions  $p$  does not belong to  $B_L \cup B_R$  and for each  $A'' \subseteq A$  it holds that in election  $(C \cup A'', V)$  the candidates from  $B_R$  receive 0 points each.

We say that a set  $Y \subseteq C \cup A$  is well-formed if for each candidate  $d \in C$  it holds that  $d \in \text{before}(Y) \cup Y \cup \text{after}(Y)$  (in other words,  $Y$  is well-formed if it is an interval with respect to the societal axis  $L$  restricted to  $C \cup Y$ ). We define  $K_L$  to be a collection of subsets of  $\text{before}(\{p\})$  such that if  $Y \in K_L$  then  $Y \cup \{p\}$  is well-formed and  $\|Y\| = k$ . (Intuitively,  $K_L$  is the family of sets of candidates that can form the “left part” of  $p$ 's neighborhood.) We define  $K_R$  analogously, but “to the right of  $p$ .” That is,  $K_R$  is a collection of subsets of  $\text{after}(\{p\})$  such that if  $Y \in K_R$  then  $\{p\} \cup Y$  is well-formed and  $\|Y\| = k$ . Note that there are at most  $O(\|C \cup A\|^k)$  sets in each of  $K_L$  and  $K_R$ .

Our algorithm proceeds as follows. In a loop, we try each  $Y_L \in K_L$  and each  $Y_R \in K_R$ . For each choice of  $Y_L$  and  $Y_R$ , we set  $Y = Y_L \cup \{p\} \cup Y_R$  to be the neighborhood of  $p$ , and we set  $\rho$  to be the  $k$ -Approval score of  $p$  in election  $(Y, V)$ . (If  $\|Y \cap A\| > \ell$  then we drop this choice of  $Y_L$  and  $Y_R$  because picking this neighborhood requires adding more candidates than we are allowed to.) We create two new sets,  $C' = C \cup Y$ ,  $A' = A \cap (\text{before}(Y) \cup \text{after}(Y))$ , and an integer  $\ell' = \ell - \|Y \cap A\|$ . Finally, we compute the number  $s(Y)$  of size-at-most- $\ell'$  subsets  $A''$  of  $A'$  such that in election  $(C' \cup A'', V)$  each candidate has less than  $\rho$  points (except for  $p$  who, by Lemma 4.7, has exactly  $\rho$  points). Computing  $s(Y)$  in polynomial time is a crucial technical part of the algorithm and we describe it below. We sum up all the values  $s(Y)$  and return them as our output. By Lemma 4.7 it is easy to see that this strategy is correct.

We now describe how to compute  $s(Y)$  in polynomial time. Let us consider a situation where we have picked  $p$ 's neighborhood  $Y$  and computed  $\rho$ ,  $C'$ ,  $A'$ , and  $\ell'$ . If  $\rho = 0$  then  $p$  cannot be the unique winner so we assume that  $\rho > 0$ . From now on, we assume that  $C'$  and  $A'$  take the roles of  $C$  and  $A$  in the definition of a well-formed set.



For each well-formed set  $Z = \{z_1, \dots, z_{2k}\} \subseteq C' \cup A'$ , such that  $z_1 L \dots L z_{2k}$ , and for each nonnegative integer  $s$  we define:

$f(Z, s)$  = the number of sets  $A'' \subseteq (A' \cap \text{before}(Z))$  such that  $\|A''\| \leq s$  and in election  $(Z \cup A'' \cup (C' \cap \text{before}(Z)))$  each candidate in  $\{z_1, \dots, z_k\} \cup A'' \cup (C' \cap \text{before}(Z))$  other than  $p$  (if  $p$  is included in this set) has fewer than  $\rho$  points.<sup>8</sup>

It is easy to see that  $s(Y)$  is simply  $f(B_R, \ell')$  (recall that  $B_R$  is the set of “right-hand side” dummy candidates, who have score 0 in every election). For each  $Z$  and  $s$ , we can compute  $f(Z, s)$  using dynamic programming. To provide appropriate recursive expression for  $f$ , we need some additional notation. We define  $\text{prev}(Z)$  to be the last candidate from  $C'$  with respect to  $L$  that precedes the candidates from  $Z$ . That is  $\text{prev}(Z)$  is the maximal (“rightmost”) element of  $C' \cap \text{before}(Z)$  with respect to  $L$ . We define  $\text{Prev}(Z)$  to be the set that contains  $\text{prev}(Z)$  and all the candidates from  $C' \cup A'$  that are between  $\text{prev}(Z)$  and  $Z$ , with respect to  $L$ . For a well-formed set  $Z = \{z_1, \dots, z_{2k}\}$  and an element  $z \in \text{Prev}(Z)$  (provided that  $\text{Prev}(Z)$  is defined) we let  $\delta(Z, z)$  be 1 if in election  $(\{z, z_1, \dots, z_{2k}\}, V)$  candidate  $z_k$  is either  $p$  or obtains fewer than  $\rho$  points. Otherwise we set  $\delta(Z, z) = 0$ . Now, for each nonnegative integer  $s$  and each well-formed set  $Z = \{z_1, \dots, z_{2k}\} \subseteq C' \cup A'$  such that  $z_1 L \dots L z_{2k}$  the following relation holds ( $\chi_{A'}$  is the characteristic function of set  $A'$ , i.e.,  $\chi_{A'}(z)$  is 1 if  $z \in A'$  and is 0 otherwise):

$$f(Z, s) = \begin{cases} \sum_{z \in \text{Prev}(Z)} \delta(Z, z) f(\{z, z_1, \dots, z_{2k-1}\}, s - \chi_{A'}(z)), & \text{if } \text{Prev}(Z) \text{ is defined,} \\ 1, & \text{otherwise.} \end{cases}$$

Note that if  $\text{Prev}(Z)$  is not defined then  $Z = B_L$  and by single-peakedness of the election,  $z_1, \dots, z_k$  have zero points each.

The correctness of this recursive expression follows by our assumptions and by Lemma 4.7. It is easy to see that using dynamic programming and this recursive expression we can compute  $f(B_R, \ell')$  in polynomial time. This concludes the proof.  $\square$

**Voter Control.** We now turn to the case of voter control and we start with the unrestricted case. This time, we need quite a few new ideas: Under  $k$ -Approval (for fixed  $k$ ,  $k \geq 2$ ) all types of counting voter control are hard, while the complexity of decision variants is quite varied (see the works of Lin [33, 13] and of Faliszewski, Hemaspaandra, and Hemaspaandra [20]).

We start by considering 2-Approval-#CCAV and 2-Approval-#CCDV separately, then we extend these results to  $k \geq 3$ , and finally we invoke Theorem 3.2 to obtain the results for the destructive cases.

**Theorem 4.8.** *In the unrestricted case, 2-Approval-#CCAV is #P-Turing-complete and 2-Approval-#CCDV is #P-metric-complete.*

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<sup>8</sup>Yes, we really mean the condition on scores to apply to candidates  $z_1, \dots, z_k$  but not to candidates  $z_{k+1}, \dots, z_{2k}$ ; we will implicitly ensure that the scores of  $z_{k+1}, \dots, z_{2k}$  do satisfy the condition as well.

*Proof.* We first consider 2-Approval-#CCAV. We give a Turing reduction from #Perfect-Matching to 2-Approval-#CCAV. Let  $G = (G(X), G(Y), G(E))$  be our input bipartite graph, where  $G(X) = \{x_1, \dots, x_n\}$  and  $G(Y) = \{y_1, \dots, y_n\}$  are sets of vertices, and  $G(E) = \{e_1, \dots, e_m\}$  is the set of edges. We form an election  $E = (C, V)$  and a collection  $W$  of unregistered voters as follows. We set  $C = \{p, b_1, b_2\} \cup G(X) \cup G(Y)$  and we let  $V = (v_1, v_2)$ , where  $v_1$  has preference order  $p > b_1 > C - \{p, b_1\}$  and  $v_2$  has preference order  $p > b_2 > C - \{p, b_2\}$ . We let  $W = (w_1, \dots, w_m)$ , where for each  $\ell$ ,  $1 \leq \ell \leq m$ , if  $e_\ell = \{x_i, y_j\}$  then  $w_\ell$  has preference order  $x_i > y_j > C - \{x_i, y_j\}$ .

Thus, in election  $E$  candidate  $p$  has score 2, candidates  $b_1$  and  $b_2$  have score 1, and candidates in  $G(X) \cup G(Y)$  have score 0. We form an instance  $I$  of 2-Approval-#CCAV with election  $E = (C, V)$ , collection  $W$  of unregistered voters, designated candidate  $p$ , and the number of voters that can be added set to  $n$ . We form instance  $I'$  to be identical, except we allow to add at most  $n - 1$  voters.

It is easy to verify that the number of 2-Approval-#CCAV solutions for  $I$  (for  $I'$ ) is the number of matchings in  $G$  of cardinality at most  $n$  (the number of matchings in  $G$  of cardinality at most  $n - 1$ ). (Each unregistered voter corresponds to an edge in  $G$  and one cannot add two edges that share a vertex as then  $p$  would no longer be the unique winner.) The number of perfect matchings in  $G$  is exactly the number of solutions for  $I$  minus the number of solutions for  $I'$ .

Let us now consider the case of 2-Approval-#CCDV. We give a metric reduction from #PerfectMatching. As before, let  $G = (G(X), G(Y), G(E))$  be our input bipartite graph, where  $X = \{x_1, \dots, x_n\}$  and  $Y = \{y_1, \dots, y_n\}$  are sets of vertices, and  $E = \{e_1, \dots, e_m\}$  is the set of edges. For each vertex  $v$  of  $G$ , we write  $d(v)$  to denote  $v$ 's degree. We set  $D = \max\{d(v) \mid v \in X \cup Y\}$  and we set  $T = \sum_{v \in X \cup Y} (D - d(v))$ . W.l.o.g. we assume that  $D \geq 2$ . We form an election  $E = (C, V)$  as follows. We set  $C = \{p\} \cup X \cup Y \cup B$ , where  $B = \{b_1, \dots, b_{T+D}\}$ . We form the collection  $V = V_E + A_{X,Y} + A_p$  of voters as follows:

1. We set  $V_E = (v_1, \dots, v_m)$  and for each  $e_\ell = \{x_i, y_j\} \in E$ , we set the preference order of  $v_\ell$  to be  $x_i > y_j > C - \{x_i, y_j\}$ .
2. We set  $A_{X,Y} = (a_1, \dots, a_T)$  and we set their preference orders so that for each  $v \in X \cup Y$ ,  $A_{X,Y}$  contains exactly  $D - d(v)$  voters with preference orders of the form  $v > b_t > C - \{v, b_t\}$  (see Item 4. below regarding how candidates  $b_t$  are chosen).
3. We set  $A_p = \{a_{T+1}, \dots, a_{T+D}\}$ , where each voter  $a_{T+i}$ ,  $1 \leq i \leq D$ , has preference order of the form  $p > b_t > C - \{p, b_t\}$  (see Item 4. below regarding how candidates  $b_t$  are chosen).
4. We arrange the preference orders of voters in  $A_{X,Y} + A_p$  so that each candidate  $b_t$ ,  $1 \leq t \leq T + D$ , receives exactly 1 point.

Thus, before deleting any of the voters, each candidate in  $\{p\} \cup X \cup Y$  has score  $D \geq 2$  and each candidate in  $B$  has score 1. We form instance  $I$  of 2-Approval-#CCDV consisting of  $E = (C, V)$ , designated candidate  $p$ , and with  $n$  as the limit on the number of voters we are allowed to delete.

We claim that the number of solutions for  $I$  is equal to the number of perfect matchings in  $G$ . Let  $M \subseteq E$  be a perfect matching in  $G$ . We form collection  $V' = \{v_i \mid e_i \in M\}$ . Clearly,  $\|V'\| \leq n$  and in election  $E' = (C, V - V')$  candidate  $p$  is the unique winner ( $p$  has  $D$  points, candidates in  $X \cup Y$  have  $D - 1$  points, and candidates in  $B$  have 1 point).

On the other hand, let  $V'$  be a subcollection of  $V$  such that  $\|V'\| \leq n$  and  $p$  is the unique winner of election  $E' = (C, V - V')$ . Since  $p$  is the unique winner of  $E'$ , it must hold that each of the  $2n$  candidates in  $X \cup Y$  has at most  $D - 1$  points in  $E'$ . Thus, since  $|V'| \leq n$ , it must be the case that in fact  $\|V'\| = n$  and  $V'$  is a subcollection of  $V_E$  such that each candidate from  $X$  appears in the first position of exactly one vote in  $V'$  and each candidate from  $Y$  appears in the second position of exactly one vote from  $V'$ . As a result,  $E' = \{e_i \mid v_i \in V'\}$  is a perfect matching for  $G$ .  $\square$

**Theorem 4.9.** *In the unrestricted case, for each  $k \geq 3$ ,  $k$ -Approval-#CCAV and  $k$ -Approval-#CCDV are #P-metric-complete.*

*Proof.* The proof for the case of #CCDV follows by applying a natural padding argument in the reduction from the proof of Theorem 4.8. Thus we focus on the case of #CCAV.

We first give a proof for the case  $k = 3$ . We give a Turing reduction from #PerfectMatching to 3-Approval-#CCAV. Let  $G = (G(X), G(Y), G(E))$  be our input bipartite graph, where  $G(X) = \{x_1, \dots, x_n\}$  and  $G(Y) = \{y_1, \dots, y_n\}$  are sets of vertices, and  $G(E) = \{e_1, \dots, e_m\}$  is the set of edges. We form an election  $E = (C, V)$  and a collection  $W$  of unregistered voters as follows. We set  $C = \{p, d\} \cup B \cup G(X) \cup G(Y)$  (where  $B$  is a collection of dummy candidates to be specified later) and we set  $V = (v_1, \dots, v_t)$ , where the preference orders of the voters in  $V$  are such that the score of  $p$  is 0, the score of  $d$  is  $n - 1$ , the score of each candidate in  $G(X) \cup G(Y)$  is  $n - 2$ , and the score of each dummy candidate in  $B$  is either 1. We set  $W = (w_1, \dots, w_m)$ , where for each  $\ell$ ,  $1 \leq \ell \leq m$ , if  $e_\ell = \{x_i, y_j\}$  then  $w_\ell$  has preference order  $p > x_i > y_j > C - \{p, x_i, y_j\}$ .

We form an instance  $I$  of 3-Approval-#CCAV with election  $E = (C, V)$ , collection  $W$  of unregistered voters, designated candidate  $p$ , and the number of voters that can be added set to  $n$ .

It is easy to verify that the number of 3-Approval-#CCAV solutions for  $I$  is the number of matchings in  $G$  of cardinality exactly  $n$ . (One has to add exactly  $n$  voters for  $p$  to defeat  $d$ ; each unregistered voter corresponds to an edge in  $G$  and one cannot add two edges that share a vertex as then the candidate corresponding to that vertex would have score  $n$ , and  $p$  would not be able to become the unique winner.) Thus, the number of perfect matchings in  $G$  is exactly the number of solutions for  $I$ . Further, clearly it is possible to implement our reduction in polynomial time.

The case for  $k > 3$  follows by natural padding arguments and extending the dummy-candidates set  $B$ .  $\square$

Finally, we obtain the results for the destructive cases through Theorem 3.2 (we remark that it applies to 2-Approval-#CCAV as well because Turing reductions are generalizations

of metric reductions; for the same reason in Table 1 we report the complexity for  $k$ -Approval-#CCAV and  $k$ -Approval-#CCDV,  $k \geq 2$ , as “#P-Turing-complete,” even though our results for  $k \geq 3$  are slightly more precise).

**Corollary 4.10.** *In the unrestricted case, 2-Approval-#DCAV is #P-Turing-complete and for each  $k$ ,  $k \geq 2$ ,  $(k + 1)$ -Approval-#DCAV and  $k$ -Approval-#DCDV are #P-metric-complete.*

Let us now move on to the single-peaked case. Here we show polynomial-time algorithms for all counting voter control problems under  $k$ -Approval (for fixed  $k$ ). Our algorithm is based on dynamic programming. We use the following notation in the proof below: For a given  $j$ -element integer vector  $scores$  and integer  $i$ ,  $1 \leq i \leq j$ , we write  $scores_i$  to denote the  $i$ -th element of vector  $scores$  (i.e.,  $scores = (scores_1, \dots, scores_j)$ ).

**Theorem 4.11.** *For the case of single-peaked profiles, for each fixed positive integer  $k$ ,  $k$ -Approval-#CCAV,  $k$ -Approval-#CCDV,  $k$ -Approval-#DCAV and  $k$ -Approval-#DCDV are in FP.*

*Proof.* We give the proof for  $k$ -Approval-#CCAV. The result for  $k$ -Approval-#CCDV follows by Theorem 3.3 and the results for the destructive cases follow by Theorem 3.2.

We are given a candidate set  $C$ , a voter collection  $V$ , a designated candidate  $p \in C$ , a collection  $W$  of unregistered voters, a positive integer  $\ell$ , and a societal axis  $L$ . We show a polynomial time algorithm for determining the number of sets  $W'$ ,  $W' \subseteq W$ , where  $\|W'\| \leq \ell$ , such that  $p$  is a  $k$ -Approval winner in election  $(C, V \cup W')$ .

For each voter  $v \in V \cup W$ , we define  $\text{top}_k(v)$  to be the set of  $k$  most preferred candidates according to  $v$ . Let  $\tilde{L}'$  be a weak linear order over voter set  $W \cup V$ , such that  $v_1 \tilde{L}' v_2$ ,  $v_1, v_2 \in V \cup W$ , if and only if there exists a candidate  $c \in \text{top}_k(v_2)$  such that for each  $c' \in \text{top}_k(v_1)$  we have  $c' L c$  or  $c' = c$ . Let  $L'$  be a strict order obtained from  $\tilde{L}'$  by breaking the ties in an arbitrary fashion. For each voter  $v \in V \cup W$  let  $X^v = \{w \mid w \in V \cup W \wedge w L' v\}$  be the subset of voters from  $V \cup W$  that precede voter  $v$  under  $L'$ . We let  $v_l$  to be the last voter from  $V$  under  $L'$ . For a given voter  $v \in V \cup W$ , integer  $z \in \mathbb{Z}$ , and integer  $s \in \mathbb{Z}$ , we define  $g(v, z, s)$  to be the number of sets  $\tilde{W} \subseteq X^v \cap W$ , where  $\|\tilde{W}\| = s$ , such that  $p$  is a  $k$ -Approval winner in election  $(C, (X^v \cap V) \cup \tilde{W} \cup \{v\})$  and in addition the score of candidate  $p$  in this election is exactly  $z$ . Equation (1) below gives the result that we should output on the given input instance of  $k$ -Approval-#CCAV problem:

$$\sum_{0 \leq s \leq \ell} \left( \sum_{0 \leq z \leq \|W \cup V\|} \left( g(v_l, z, s) + \sum_{\substack{w \in W \\ v_l L' w}} g(w, z, s) \right) \right) \quad (1)$$

Thus, it suffices to show how to compute  $g(w, z, s)$  for  $w \in V \cup W$  and  $z, s \in \mathbb{Z}$  in polynomial time. Before we give an appropriate algorithm, we need to introduce some additional notation.

For each candidate  $c \in C$ , let  $\text{rank}(c)$  be  $c$ 's rank under  $L$  over all candidates from  $C$  (so, for example, if  $C = \{a, b, c\}$  and  $a L b L c$  then  $\text{rank}(a) = 1$ ,  $\text{rank}(b) = 2$  and  $\text{rank}(c) = 3$ ). For each voter  $v$ , let  $\text{rank}(v) = \max\{\text{rank}(c) \mid c \in \text{top}_k(v)\}$ . For each  $v \in V \cup W$ , let  $\mathcal{P}^v = \{w \mid w \in (V \cup W) \wedge w L' v \wedge (\nexists t \in V)[w L' t L' v]\}$ . In other words,  $\mathcal{P}^v$  consists of the closest voter  $u \in V$  that precedes  $v$  under  $L'$ , and of all the voters that are between  $u$  and  $v$  under  $L'$ . When  $v$  is the first element from  $V$  under  $L'$ , then  $\mathcal{P}^v$  contains all the voters from  $W$  preceding  $v$  under  $L'$ .

For a  $j$ -element integer vector  $\text{scores}$  and a nonnegative integer  $r$ , let  $\text{cutoff}(\text{scores}, r)$  denote  $j$ -element vector  $(\text{scores}_1, \dots, \text{scores}_r, 0, \dots, 0)$  (that is, we replace the last  $j - r$  entries of vector  $\text{scores}$  with zeros). For each voter  $w \in V \cup W$ , we define  $\text{approval}(w)$  to be the  $\|C\|$ -dimensional 0/1 vector that for each candidate  $c \in C$  has 1 at position  $\text{rank}(c)$  if and only if  $c \in \text{top}_k(w)$ . (In other words,  $\text{approval}(w)$  is the 0/1 vector that indicates which candidates receive points from voter  $w$ .) Note that, due to single-peakedness of the election, for each voter  $w$ ,  $\text{approval}(w)$  contains exactly a single consecutive block of  $k$  ones.

We are now ready to proceed with our algorithm for computing function  $g$ . For a given voter  $v \in V \cup W$ , given integers  $z, s \in \mathbb{Z}$ , and a  $\|C\|$ -dimensional integer vector  $\text{scores}$ , we define  $f(v, z, s, \text{scores})$  to be the number of sets  $\widetilde{W} \subseteq X^v \cap W$  such that  $\|\widetilde{W}\| = s$  and in election  $(C, (X^v \cap V) \cup \widetilde{W} \cup \{v\})$  the following holds: (1)  $p$  scores exactly  $z$  points and (2) each candidate  $c \in C - \{p\}$  scores no more than  $\text{scores}_{\text{rank}(c)}$  points. For a given integer  $r \in \mathbb{Z}$ , let  $\Gamma^r$  be the vector  $(r, \dots, r)$  of dimension  $\|C\|$ . Clearly, for a given voter  $v \in V \cup W$  and given integers  $z, s \in \mathbb{Z}$  we have:

$$g(v, z, s) = f(v, z, s, \Gamma^{z-1}) \quad (2)$$

We now show a recursive formula for  $f$ . For a given voter  $v \in V \cup W$ , a given vector  $\text{scores}$  of (nonnegative) integers of dimension  $\|C\|$ , and two integers  $z, s \in \mathbb{Z}$ , we have:

$$f(v, z, s, \text{scores}) = f(v, z, s, \text{cutoff}(\text{scores}, \text{rank}(v))) \quad (3)$$

This follows from the fact that in election  $(C, X^v \cup \{v\})$  only candidates with ranks  $j \leq \text{rank}(v)$  can score a point. It is easy to see that  $f(v, z, s, \text{scores}) = 0$  when  $z < 0$  or  $\text{scores}$  contains at least one negative entry, because the score is always a nonnegative integer. When  $s = 0$  then  $f(v, z, s, \text{scores})$  is 1 if and only if in election  $(C, (X^v \cap V) \cup \{v\})$  candidate  $p$  scores  $z$  points and each candidate  $c \in C$  scores no more than  $\text{scores}_{\text{rank}(c)}$  points; otherwise  $f(v, z, s, \text{scores})$  is 0. When  $s > 0$ , we note that each election consistent with the condition for  $f(v, z, s, \text{scores})$  contains at least one voter  $w$  from  $\mathcal{P}^v$ . Eq. (4) below gives formula for  $f$  in case when  $s > 0$ ; for each voter  $w \in \mathcal{P}^v$  we count all the sets  $\widetilde{W} \subseteq X^v \cap W$  such that  $w$  directly precedes  $v$  in  $(X^v \cap V) \cup \widetilde{W} \cup \{v\}$  under  $L'$ :

$$f(v, z, s, \text{scores}) = \sum_{w \in \mathcal{P}^v} f(w, z - z', s - s', \text{scores} - \text{approval}(w)), \quad (4)$$

where (1)  $z' = 1$  if  $p \in \text{top}_k(v)$  and  $z' = 0$  otherwise, and (2)  $s' = 1$  if  $v \in W$  and  $s' = 0$

otherwise. By combining Eq. (3) and (4) we get:

$$f(v, z, s, scores) = \sum_{w \in \mathcal{P}_v} f(w, z - z', s - s', \text{cutoff}(scores - \text{approval}(w), \text{rank}(w))). \quad (5)$$

We claim that function  $g$  can be computed through Eq. (2) in polynomial time, using standard dynamic programming techniques to compute function  $f$ . The reason is that to compute  $f$  for the arguments as in Eq. (2) using recursive formula (5), we need to obtain  $f$ 's values for at most  $\|C\|(\|V\| + \|W\|)^{k+3}$  different arguments. This is because starting from  $scores = \Gamma^{z-1}$ , the only allowed transformations of  $scores$  are given in Eq. (5) and ensure that whenever we need to compute  $f$ , the  $scores$  vector is of the form  $(z - 1, z - 1, \dots, z - 1, e_1, e_2, \dots, e_k, 0, 0, \dots, 0)$ , where  $e = (e_1, e_2, \dots, e_k)$  is some  $k$ -element vector of integers and for each  $i$ ,  $1 \leq i \leq k$ , we have  $0 \leq e_i \leq z - 1$ . Clearly, there are no more than  $\|C\|z^k$  vectors of this form. Thus, we can compute  $g$  in polynomial time and, through Eq. (1), we can solve  $k$ -Approval-#CCAV in polynomial time.  $\square$

### 4.3 Approval Voting and Condorcet Voting

Let us now consider Approval voting and Condorcet voting. While these two systems may seem very different in spirit, their behavior with respect to election control is similar. Specifically, in the unrestricted case, for both systems #CCAV and #CCDV are #P-complete, for both systems it is impossible to make some candidate a winner by adding candidates, and for both systems it is impossible to prevent someone from winning by deleting candidates. Yet, for both systems #DCAC and #CCDC are in FP via almost identical algorithms. There is, however, also one difference. For the single-peaked case, we were able to find polynomial-time algorithms for voter control under Condorcet, while the results for Approval remain elusive.

**Theorem 4.12.** *In the unrestricted case, each of Approval-#CCAV, Approval-#CCDV, Condorcet-#CCAV, and Condorcet-#CCDV is #P-complete. Their destructive variants are #P-metric-complete.*

*Proof.* Note that the results for the destructive variants will follow by Theorem 3.2 and so we focus on constructive variants only.

For the case of Approval, #P-completeness of #CCAV and #CCDV follows from the NP-completeness proofs for their decision variants given by Hemaspaandra, Hemaspaandra, and Rothe [28]; these proofs, in effect, give parsimonious reductions from #X3C to respective control problems and, thus, establish #P-completeness.

For the case of Condorcet, #P-completeness of #CCDV follows from the proofs of Theorems 5.1 and 4.19 of Faliszewski et al. [22], who effectively give a parsimonious reduction from #X3C to Condorcet-#CCDV. The case of Condorcet-#CCAV appears to not have been considered in the literature and thus we give a direct proof (naturally, we could obtain #P-Turing-completeness by noting that Theorem 3.3 gives a Turing reduction from Condorcet-#CCDV to Condorcet-#CCAV, but #P-completeness is a stronger result). For

the remainder of the proof we focus on Condorcet-#CCAV. The problem is clearly in #P, so it suffices to show that it is #P-hard. We give a parsimonious reduction from #X3C.

Let  $(B, \mathcal{S})$  be an instance of #X3C problem, where  $B = \{b_1, \dots, b_{3k}\}$  and  $\mathcal{S} = \{S_1, \dots, S_n\}$ . We create an election  $E = (C, V)$ , where  $C = B \cup \{p\}$ . Let  $V$  consist of  $k - 3$  voters with preferences  $b_1 \succ b_2 \succ \dots \succ b_{3k} \succ p$ . Thus,  $b_1$  is the Condorcet winner of  $E$ , and every candidate  $b_i \in B$  beats  $p$  in  $k - 3$  votes.

For each set  $S_j \in \mathcal{S}$ ,  $S_j = \{b_{j_1}, b_{j_2}, b_{j_3}\}$ , let  $W$  contain a voter with preference order  $b_{j_1} \succ b_{j_2} \succ b_{j_3} \succ p \succ \dots$  (after  $p$  the remaining candidates are ranked in arbitrary order). We claim that every subset  $W'$ ,  $W' \subseteq W$ , such that  $\|W'\| \leq k$  and that  $p$  is a Condorcet winner of election  $E' = (C, V \cup W')$  corresponds one-to-one to a  $k$ -element subfamily  $\mathcal{S}'$  of  $\mathcal{S}$  whose elements union up to  $B$  (i.e.,  $\mathcal{S}'$  is an exact set cover of  $B$ ).

First assume that there is a subfamily  $\mathcal{S}' \subseteq \mathcal{S}$  which is an exact set cover of  $B$ . For each  $S_j \in \mathcal{S}'$ , we include the corresponding voter from  $W$  in  $W'$ . Let us consider election  $E' = (C, V \cup W')$ . For each  $b_i \in B$  we have  $N_{E'}(b_i, p) = N_E(b_i, p) + 1 = k - 2$  and  $N_{E'}(p, b_i) = N_E(p, b_i) + k - 1 = k - 1$ . Thus  $p$  becomes the Condorcet winner of  $E'$ .

Now assume that  $p$  is the Condorcet winner in election  $E' = (C, V \cup W')$ , where  $W' \subseteq W$  and  $\|W'\| \leq k$ . For each  $b_i \in B$ , there can be at most one voter in  $W'$  that prefers  $b_i$  to  $p$ . This is so, because otherwise we would have  $N_{E'}(b_i, p) \geq N_E(b_i, p) + 2 = k - 1$ , and  $N_{E'}(p, b_i) \leq N_E(p, b_i) + k - 2 = k - 2$ , and so  $p$  would lose to  $b_i$ . Thus, each  $b_i$  is preferred to  $p$  by either zero or one voter from  $W'$ . If  $b_i$  is preferred by one voter from  $W'$ , then for  $p$  to win,  $p$  must be preferred to  $b_i$  by  $k - 1$  voters from  $W'$ , and since some voter must be added, it must hold that  $\|W'\| = k$ .

If there are no voters in  $W'$  who prefer  $b_i$  to  $p$ , then since each vote in  $W'$  has some three candidates from  $B$  ranked ahead of  $p$ , there must be some other  $b_{i'}$  that is ranked above  $p$  by more than one voter. This contradicts the requirement that for each  $b_j \in B$ , at most one voter in  $W'$  prefers  $b_j$  to  $p$ . Hence, each  $b_i$  is preferred to  $c$  by exactly one of the  $k$  voters in  $W'$ . Thus, the voters from  $W'$  correspond to an exact set cover of  $B$ .  $\square$

For the case of single-peaked preferences, we can use dynamic programming to solve voter control problems under Condorcet.

**Theorem 4.13.** *For the single-peaked case, Condorcet-#CCAV, Condorcet-#CCDV, Condorcet-#DCAV, and Condorcet-#DCDV are in FP.*

*Proof.* We focus on Condorcet-#CCAV. The remaining cases follow by applying Theorems 3.2 and 3.3.

We are given an election  $E = (C, V)$  where  $C$  is a set of candidates and  $V$  is a set of voters, a designated candidate  $p \in C$ , a collection  $W$  of unregistered voters, a nonnegative integer  $k$ , and an order  $L$ , the societal axis, such that  $V$  and  $W$  are single-peaked with respect to  $L$ . We show a polynomial-time algorithm for determining the number of sets  $W'$ ,  $W' \subseteq W$ , where  $\|W'\| \leq k$ , such that  $p$  is a Condorcet winner in election  $(C, V \cup W')$ . Our algorithm is based on the famous median-voter theorem.

We split  $C$  into three sets,  $C_a = \{c \mid c \in C \wedge c \prec_L p\}$ ,  $C_b = \{c \mid c \in C \wedge p \prec_L c\}$  and  $C_m = \{p\}$ ;  $C_a$  contains the candidates that are before  $p$  on the societal axis and  $C_b$  contains

the candidates that are after  $p$ . For each voter  $v$ , by  $c_v$  we mean the candidate that  $v$  ranks first. For each collection  $U$  of voters from  $V \cup W$ , we define  $U_a = \{v \mid v \in U \wedge c_v \in C_a\}$ ,  $U_b = \{v \mid v \in U \wedge c_v \in C_b\}$  and  $U_m = \{v \mid v \in U \wedge c_v = p\}$ . In other words,  $U_a$  and  $U_b$  consist of those voters from  $U$  for which the most preferred candidate is, respectively, in  $C_a$  or  $C_b$ , and  $U_m$  contains those voters from  $U$  that rank  $p$  first.

For each  $W' \subseteq W$ , we define  $\delta(W')$  to be 1 exactly if the following conditions hold:

1.  $\|V_a\| + \|W'_a\| < \frac{1}{2}(\|V\| + \|W'\|)$ , and
2.  $\|V_b\| + \|W'_b\| < \frac{1}{2}(\|V\| + \|W'\|)$ .

Otherwise, we define  $\delta(W') = 0$ . The following lemma is an expression of the well-known median voter theorem (we provide the short proof for the sake of completeness).

**Lemma 4.14.** *For each  $W' \subseteq W$ ,  $p$  is the Condorcet winner of election  $(C, V \cup W')$  if and only if  $\delta(W') = 1$ .*

*Proof.* Assume that  $\delta(W') = 1$  and consider an arbitrary candidate  $c$  other than  $p$ . For the sake of concreteness let us assume that  $c \in C_a$ , but a symmetric argument holds for  $c \in C_b$ . Due to single-peakedness of  $V \cup W'$ , no voter outside of  $V_a$  and  $W'_a$  prefers  $c$  to  $p$ . However, since  $\delta(W') = 1$ , we have that  $\|V_a\| + \|W'_a\| < \frac{1}{2}(\|V\| + \|W'\|)$ , and so a majority of the voters prefers  $p$  to  $c$ . Since  $c$  was chosen arbitrarily, it holds that  $p$  is a Condorcet winner.

For the other direction, assume that  $\delta(W') = 0$ . For the sake of concreteness, let us assume that this is because  $\|V_a\| + \|W'_a\| \geq \frac{1}{2}(\|V\| + \|W'\|)$ . By this assumption, there must be a candidate  $c \in C$  that directly precedes  $p$  with respect to  $L$  (that is,  $c L p$  and there is no candidate  $d$  such that  $c L d L p$ ). Due to single-peakedness of voters' preference orders, we have that every voter in  $V_a \cup W'_a$  prefers  $c$  to  $p$ , and so  $p$  is not a Condorcet winner. A symmetric argument holds if  $\|V_b\| + \|W'_b\| \geq \frac{1}{2}(\|V\| + \|W'\|)$ .  $\square$

Thus our algorithm should output the number of sets  $W'$ ,  $W' \subseteq W$ , of cardinality at most  $k$ , such that  $\delta(W') = 1$  holds. However, to evaluate  $\delta(W')$  it suffices to know the values  $\|W'_a\|$ ,  $\|W'_b\|$ , and  $\|W'_m\|$ . For each three integers  $\ell_a$ ,  $\ell_b$ , and  $\ell_m$  we define  $\gamma(\ell_a, \ell_b, \ell_m)$  to be 1 exactly if the following two conditions hold (these conditions are analogous to those for  $\delta$ ):

1.  $\|V_a\| + \ell_a < \frac{1}{2}(\|V\| + \ell_a + \ell_b + \ell_m)$ , and
2.  $\|V_b\| + \ell_b < \frac{1}{2}(\|V\| + \ell_a + \ell_b + \ell_m)$ .

Otherwise,  $\gamma(\ell_a, \ell_b, \ell_m) = 0$ . It is easy to see that for each  $W' \subseteq W$  we have  $\delta(W') = \gamma(\|W'_a\|, \|W'_b\|, \|W'_m\|)$ . Now it follows that the number of subsets  $W' \subseteq W$  of cardinality at most  $k$  such that  $p$  is a Condorcet winner of election  $(C, V \cup W')$  is exactly:

$$\sum_{\ell=0, \dots, k} \sum_{\substack{\ell_a, \ell_b, \ell_m \in \mathbb{N} \\ \ell_a + \ell_b + \ell_m = \ell}} \gamma(\ell_a, \ell_b, \ell_m) \binom{\|W_a\|}{\ell_a} \binom{\|W_b\|}{\ell_b} \binom{\|W_m\|}{\ell_m}.$$

We can evaluate this sum in polynomial-time. This completes the proof.  $\square$



For the case of candidate control, we get polynomial-time algorithms even for the unrestricted case.

**Theorem 4.15.** *Approval-#CCDC, and Condorcet-#CCDC are in FP, even in the unrestricted case.*

*Proof.* Let us handle the case of Approval first. Let  $I = (C, V, p, k)$  be an instance of approval-#CCDC. The only way to ensure that  $p \in C$  is the unique winner is to remove all candidates  $c \in C - \{p\}$  such that  $\text{score}_{(C,V)}^a(c) \geq \text{score}_{(C,V)}^a(p)$ . Such candidates can be found immediately. Let's assume that there are  $k_0$  such candidates. After removing all of them, we can also remove  $k - k_0$  or less of any remaining candidates other than  $p$ . Based on this observation we provide the following simple algorithm.

APPROVAL-#CCDC( $C, V, p, k$ )

- 1 Let  $k_0$  be the number of candidates  $c \in C - \{p\}$ ,  
s.t.  $\text{score}_{(C,V)}^a(c) \geq \text{score}_{(C,V)}^a(p)$ .
- 2 **return**  $\sum_{i=0}^{k-k_0} \binom{\|C\| - k_0 - 1}{i}$

Clearly, the algorithm is correct and runs in polynomial-time.

For the case of Condorcet voting, it suffices to note that if  $p$  is to be a winner, we have to delete all candidates  $c \in C - \{p\}$  such that  $N_{(C,V)}(p, c) \leq N_{(C,V)}(c, p)$ . Thus, provided that we let  $k_0$  be the number of candidates  $c \in C - \{p\}$  such that  $N_{(C,V)}(p, c) \leq N_{(C,V)}(c, p)$ , the same algorithm as for the case of approval voting works for Condorcet voting.  $\square$

**Theorem 4.16.** *Both Approval-#DCAC and Condorcet-#DCAC are in FP.*

*Proof.* We first consider the case of approval voting. Let  $I = (C, A, V, p, k)$  be an instance of Approval-#DCAC, where  $C = \{p, c_1, \dots, c_{m-1}\}$  is the set of registered candidates,  $A = \{a_1, \dots, a_{m'}\}$  is the set of additional candidates,  $V$  is the set of voters,  $p$  is the designated candidate, and  $k$  is the upper bound on the number of candidates that we can add. We will give a polynomial-time algorithm that counts the number of up-to- $k$ -element subsets  $A'$  of  $A$  such that  $p$  is not the unique winner of election  $(C \cup A', V)$ .

Let  $A_0$  be the set of candidates in  $A$  that are approved by at least as many voters as  $p$  is. To ensure that  $p$  is not the unique winner of the election (assuming  $p$  is the unique winner prior to adding any candidates), it suffices to include at least one candidate from  $A_0$ . Thus, we have the following algorithm.

APPROVAL-#DCAC( $C, A, V, p, k$ )

```

1  if  $p$  is not the unique winner of  $(C, V)$ 
2    then return  $\sum_{i=0}^k \binom{\|A\|}{i}$ 
3  Let  $A_0$  be the set of candidates  $a_i \in A$ ,
   s.t.  $\text{score}_{(C \cup A, V)}^a(a_i) \geq \text{score}_{(C \cup A, V)}^a(p)$ .
4  result := 0
5  for  $j := 1$  to  $k$ 
6    do result := result +  $\sum_{i=1}^{\min(\|A_0\|, j)} \binom{\|A_0\|}{i} \binom{\|A - A_0\|}{j-i}$ 
7  return result

```

The loop from line 5, for every  $j$ , counts the number of ways in which we can choose exactly  $j$  candidates from  $A$ ; it can be done by first picking  $i$  of the candidates in  $A_0$  (who beat  $p$ ), and then  $j - i$  of the candidates in  $A - A_0$ . It is clear that the algorithm is correct and runs in polynomial time.

Let us now move on to the case of Condorcet voting. It is easy to see that the same algorithm works correctly, provided that we make two changes: (a) in the first two lines, instead of testing if  $p$  is an approval winner we need to test if  $p$  is a Condorcet winner, and (b) we redefine the set  $A_0$  to be the set of candidates  $a_i \in A$  such that  $N_{(C \cup A, V)}(p, a_i) \leq N_{(C \cup A, V)}(a_i, p)$ . To see that these two changes suffice, it is enough to note that to ensure that  $p$  is not a Condorcet winner of the election we have to have that either  $p$  already is not a Condorcet winner (and then we can freely add any number of candidates), or we have to add at least one candidate from  $A_0$ .  $\square$

We conclude our discussion of Condorcet voting by noting that for the single-peaked case all our results for Condorcet directly translate to all Condorcet-consistent rules. The reason for this is that if voters' preferences are single peaked, then there always exist weak Condorcet winners. Whenever weak Condorcet winners exist, they are the sole winners under Condorcet-consistent rules by definition. Since we focus on the unique-winner model, if  $R$  is a Condorcet-consistent rule,  $E$  is a single-peaked, and  $c$  is a candidate, then  $c$  is the unique  $R$ -winner of  $E$  if and only if  $c$  is the unique Condorcet winner of  $E$ . In effect, we have the following corollary.

**Corollary 4.17.** *Let  $R$  be a Condorcet-consistent rule. For the single-peaked case,  $R$ -#CCDC,  $R$ -#DCAC,  $R$ -#CCAV,  $R$ -#CCDV,  $R$ -#DCAV, and  $R$ -#DCDV are in FP.*

#### 4.4 Maximin Voting

For the case of Maximin, we consider the unrestricted case only. Maximin is Condorcet-consistent and, thus, for the single-peaked case we can use Corollary 4.17.

The complexity of decision variants of control for Maximin was studied by Faliszewski, Hemaspaandra, and Hemaspaandra [19]. In particular, they showed that under Maximin all voter control problems are NP-complete and an easy adaptation of their proofs gives the following theorem.

**Theorem 4.18.** *Maximin-#CCAV and Maximin-#CCDV are #P-complete, and Maximin-#DCAV and Maximin-#DCDV are #P-metric-complete.*

On the other hand, among the candidate control problems for Maximin, only Maximin-CCAC is NP-complete (DCAC, CCDC, and DCDC are in P). Still, this hardness of control by adding candidates translates into the hardness of all the counting variants of candidate control.

**Theorem 4.19.** *Maximin-#CCAC is #P-complete and Maximin-#DCAC is #P-metric-complete.*

*Proof.* Faliszewski et al. [19]’s proof that Maximin-CCAC is NP-complete can be used without change; their reduction from X3C to Maximin-CCAC is also correct as a parsimonious reduction from #X3C to Maximin-#CCAC. The result for Maximin-#DCAC follow through Theorem 3.2.  $\square$

The cases of Maximin-#CCDC and Maximin-#DCDC are more complicated and require new ideas because decision variants of these problems are in P.

**Theorem 4.20.** *Both Maximin-#CCDC and Maximin-#DCDC are #P-Turing-complete.*

*Proof.* We consider the #CCDC case first. Clearly, the problem belongs to #P and it remains to show hardness. We will do so by giving a Turing reduction from #PerfectMatching.

Let  $G = (G(X), G(Y), G(E))$  be our input graph, where  $G(X) = \{x_1, \dots, x_n\}$  and  $G(Y) = \{y_1, \dots, y_n\}$  are sets of vertices, and  $E = \{e_1, \dots, e_m\}$  is the set of edges. For each nonnegative integer  $k$ , define  $g(k)$  to be the number of matchings in  $G$  that contain exactly  $k$  edges (e.g.,  $g(n)$  is the number of perfect matchings in  $G$ ).

We define the following election  $E = (C, V)$ . We set  $C = G(E) \cup S \cup B \cup \{p\}$ , where  $S = \{s_0, \dots, s_n\}$  and  $B = \{b_{i,j}^\ell \mid 0 \leq \ell \leq n, i < j, \text{ and } e_i \text{ and } e_j \text{ share a vertex}\}$ . To build voter collection  $V$ , for each two candidates  $a, b \in C$ , we define  $v(a, b)$  to be a pair of voters with preference orders  $a > b > C - \{a, b\}$  and  $\overleftarrow{C - \{a, b\}} > a > b$ . We construct  $V$  as follows:

1. For each  $s_i \in S$ , we add pair  $v(s_i, p)$ .
2. For each  $s_i \in S$ , we add two pairs  $v(s_i, s_{i+1})$ , where  $i + 1$  is taken modulo  $n + 1$ .
3. For each  $s_i \in S$  and each  $e_t \in E$ , we add two pairs  $v(s_i, e_t)$ .
4. For each  $e_i, e_j \in E$ ,  $i < j$ , where  $e_i$  and  $e_j$  share a vertex, and for each  $\ell$ ,  $0 \leq \ell \leq n$ , we add two pairs  $v(e_i, b_{i,j}^\ell)$  and two pairs  $v(e_j, b_{i,j}^\ell)$ .

Let  $T$  be the total number of pairs  $v(a, b)$ ,  $a, b \in C$ , included in  $V$ . By our construction, the following properties hold:

1.  $\text{score}_E^m(p) = T - 1$  and it is impossible to change the score of  $p$  by deleting  $n$  candidates or fewer (this is because there are  $n + 1$  candidates  $s_i \in S$  such that  $N_E(p, s_i) = T - 1$ ).

2. For each  $s_i \in S$ ,  $\text{score}_E^m(s_i) = T - 2$ , but deleting  $s_{i-1}$  (where we take  $i - 1$  modulo  $n + 1$ ) increases the score of  $s_i$  to  $T + 1$ .
3. For each  $e_t \in E$ ,  $\text{score}_E^m(e_t)$  is  $T - 2$ .
4. For each  $b_{i,j}^\ell \in B$ ,  $\text{score}_E^m(b_{i,j}^\ell) = T - 2$  and it remains  $T - 2$  if we delete either  $e_i$  or  $e_j$ , but it becomes  $T$  if we delete both  $e_i$  and  $e_j$ .

Note that  $p$  is the unique winner of  $E$ . For each  $k$ ,  $0 \leq k \leq n$ , we form instance  $I(k) = (C, V, p, k)$  of Maximin-#CCDC. We define  $f(k) = \#I(k) - \#I(k - 1)$ . That is,  $f(k)$  is the number of solutions for  $I(k)$  where we delete exactly  $k$  candidates. We claim that for each  $k$ ,  $1 \leq k \leq n$ , it holds that  $f(k) = \sum_{j=0}^k \binom{\|B\|}{j} g(k - j)$ . Why is this so? First, note that by the listed-above properties of  $E$ , deleting any subset  $C'$  of candidates from  $C$  that contains some member of  $S$  prevents  $p$  from being a winner. Thus, we can only delete subsets  $C'$  of  $C$  that contains candidates in  $G(E) \cup B$ . Let us fix a nonnegative integer  $r$ ,  $0 \leq r \leq n$ . Let  $C' \subseteq G(E) \cup B$  be such that  $p$  is the unique Maximin winner of  $E' = (C - C', V)$  and  $\|C'\| = r$ . Let  $r_B = \|C' \cap B\|$  and  $r_{G(E)} = \|C' \cap G(E)\|$ . It must be the case that for each  $e_i, e_j \in G(E)$ ,  $i < j$ , where  $e_i$  and  $e_j$  share a vertex,  $C'$  contains at most one of them. Otherwise,  $E'$  would contain at least one of the candidates  $b_{i,j}^\ell$ ,  $0 \leq \ell \leq n$ , and this candidate would have score higher than  $p$ . Thus, the candidates in  $C' \cap G(E)$  correspond to a matching in  $G$  of cardinality  $r_{G(E)}$ . On the other hand, since  $r_B \leq n$ ,  $C' \cap B$  contains an arbitrary subset of  $B$ . Thus, there are exactly  $\binom{\|B\|}{r_B} g(r_{G(E)})$  such sets  $C'$ . Our formula for  $f(k)$  is correct.

Now, using standard algebra (a process similar to Gauss elimination), it is easy to verify that given values  $f(1), f(2), \dots, f(n)$ , it is possible to compute (in this order)  $g(0), g(1), \dots, g(n)$ . Together with the fact that constructing each  $I(k)$ ,  $0 \leq k \leq n$ , requires polynomial time with respect to the size of  $G$ , this proves that given oracle access to Maximin-#CCDC, we can solve #PerfectMatching. Thus, Maximin-#CCDC is #P-Turing-complete and, by Theorem 3.2, so is Maximin-#DCDC.  $\square$

## 5 Related Work

The focus of this paper is on the complexity of predicting election winners for the case, where we are uncertain about the structure of the election (the exact identities of candidates/voters that participate), yet we have perfect knowledge of voters' preference orders. However, our model is just one of many approaches to winner prediction, which in various forms and shapes has been studied in the literature for some years already. For example, to model imperfect knowledge regarding voters' preferences, Konczak and Lang [30] introduced the possible winner problem, further studied by many other researchers (see, e.g., [53, 5, 1, 10, 54]). In the possible winner problem, each voter is represented via a partial preference order and we ask if there is an extension of these partial orders to total orders that ensures a given candidate's victory. Bachrach, Betzler, and Faliszewski [1] extended the model by considering counting variants of possible winner problems. Namely, they asked for how many extensions of the votes a given candidate wins, in effect obtaining the probability of

the candidate’s victory. This is very similar to our approach, but there are also important differences. In the work of Bachrach et al., we have full knowledge regarding the identities of candidates and voters participating in the election, but we are uncertain about voters’ preference orders. In our setting, we have full knowledge about voters’ preference orders, but we are uncertain about the identities of candidates/voters participating in the election.

Another model of predicting election outcomes is that of Hazon et al. [26]. They consider a situation where each voter is undecided regarding several possible votes. That is, for each voter we are given several possible preference orders and a probability distribution over these votes. The question is, what is the probability that a designated candidate wins.

From a technical standpoint, our research continues the line of work on the complexity of control. This line of work was initiated by Bartholdi, Tovey, and Trick [3], and then continued by Hemaspaandra, Hemaspaandra, and Rothe [28] (who introduced the destructive cases), by Meir et al. [37] (who considered multiwinner rules and who generalized the idea of the constructive and destructive cases), by Faliszewski, Hemaspaandra, and Hemaspaandra [19] (who introduced multimode model of control), by Faliszewski, Hemaspaandra, and Hemaspaandra [20] (who were first to consider control for weighted elections), by Rothe and Schend [46] (who initiated the empirical study of the complexity of control problems), and by many other researchers, who provided results for specific voting rules and who introduced various other novel means of studying control problems (see, e.g., the following papers [6, 15, 16, 34, 35, 38, 45]; we also point the readers to the survey [18]).

Single-peaked elections were studied for a long time in social choice literature, but they gained popularity in the computational social choice world fairly recently, mostly due to the papers of Walsh [51] and Conitzer [11]. Then, Faliszewski et al. [23] and, later, Brandt et al. [8] studied the complexity of control problems for single-peaked elections. Recently, Faliszewski, Hemaspaandra, and Hemaspaandra [21] complemented this line of work by studying nearly single-peaked profiles.

Going in a different direction, our work is very closely related to the paper of Walsh and Xia [52] on lot-based elections. Walsh and Xia study a model of Venetian elections, where a group of voters of a given size is randomly selected from a group of eligible voters, and the votes are collected from these selected voters only. In this setting, the problem of computing a candidate’s chances of victory, in essence, boils down to the counting variant of control by adding voters problem. Thus, our paper and that of Walsh and Xia are quite similar on the technical front. The papers, however, have no overlap in terms of results.

## 6 Conclusions and Future Work

We have considered a model of predicting election winners in settings where there is uncertainty regarding the structure of the election (that is, regarding the exact set of candidates and the exact collection of voters participating in the election). We have shown that our model corresponds to the counting variants of election control problems (specifically, we have focused on election control by adding/deleting candidates and voters). We have considered Plurality, Approval, Condorcet,  $k$ -Approval, and Maximin (see Table 1 for our results). For

the former three, the complexity of counting variants of control is analogous to the complexity of decision variants of respective problems, but for the latter two, some of the counting control problems are more computationally demanding than their decision counterparts.

Many of our results indicate computational hardness of winner prediction problems. To alleviate this issue to some extent, we also considered single-peaked preferences that are more likely to appear in practice. In this case, we got polynomial-time results only (except the case of Approval, where we have no results for the single-peaked case). Still, sometimes in practice one might have to seek heuristic algorithms or approximate solutions (e.g., sampling-based algorithms similar to the one of Bachrach, Betzler, and Faliszewski [1, Theorem 6]).

There are many ways to extend our work. For example, in the introduction we mentioned the model where for each voter  $v$  (or candidate  $c$ ) we have probability  $p_v$  (probability  $p_c$ ) that this voter (candidate) participates in the election. We believe that studying this problem in more detail would be very interesting. As we have argued, the model where all values  $p_v$  ( $p_c$ ) are identical, reduces to our setting, but the cases where the probabilities can differ remain open.

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